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Cauchy's stress theory in a modern light

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Abstract
The 180 year old stress theory by Cauchy is found to be insufficient to serve as a basis for a modern understanding of material behaviour. Six reasons are discussed in detail: (1) Cauchy's theory, following Euler, considers forces interacting with planes. This is in contrast to Newton's mechanics which considers forces interacting with radius vectors. (2) Bonds in solids have never been taken into account. (3) Cauchy's stress theory does not meet the minimum conditions for vector spaces because it does not have a metric. It is not a field theory, and not in the Euclidean space. (4) Cauchy's theory contains a hidden boundary condition that makes it less than general. (5) The current theory of stress is found to be at variance with the theory of potentials. (6) The theory is conceptually incompatible with thermodynamics for physical and geometrical reasons.

Keywords: deformation theory, stress, strain, elastic work

Introduction

Cauchy (1789–1857) is considered the father of modern mathematics. Apart from purely mathematical subjects, he also thought about elasticity and viscous flow. In 1823, he published a short outline of his concepts. Between 1827 and 1829, he wrote paper after paper in a series which filled the pages of his own journal. Perhaps it was this awe inspiring firework of papers that left his contemporaries speechless. I refer to his papers in time order [1–15].

Science is—or should be—the practice of asking questions; at least it is prudent to look back from time to time to check one's basis. Explanations that were found long ago may turn out to be invalid or irrelevant once the context has been better understood. Some aspects of Cauchy's views of stress and deformation of solids and fluids cannot be correct in a modern light. The full framework of classical physics became known only after 1850 and the discovery of the first law of thermodynamics. This author has outlined the loopholes before, referring to mid-20th century textbooks [16–20]. It is nonetheless worth following Cauchy's argumentation in his own writings because he is, after all, the ultimate source for the current theory.

Cauchy's theory is standard teaching matter to this day, but it is well known to cause irritation in students. In the end authority wins, but I contend that in this case the students are right; but they still react intuitively, they do not have enough knowledge yet to transform their reflex into a clear physical question. I myself asked questions to the six points below in my own introductory class in California over 30 years ago and was left unsatisfied. But young students have something the experienced practitioner no longer has: an absolutely clean slate, they can look at an old problem with perfectly innocent eyes. This essay therefore concerns the practice of everyday teaching.

Cauchy's writings

Relation to Newton's mechanics

> Citation 1: [. . .] ξ, η, ζ *désignant* [. . .] *les coordonnées du centre de gravité de la surface s. Soit maintenant m la masse infiniment petite comprise sous le volume v. Concevons en outre que la lettre* φ *représente la force accélératrice appliquée à cette masse, si le corps solide est en équilibre* [. . .]. *Enfin nommons X, Y, Z les projections algébriques de la force* φ, *et* ξ_0, η_0, ζ_0 *les coordonnées du centre de gravité de la masse m. Si l'on suppose que la force accélératrice* φ *reste la même en grandeur et en direction dans tous les points de la masse m, il devra y avoir équilibre entre la force motrice m*φ *appliquée au point* (ξ_0, η_0, ζ_0), *et les forces auxquelles se réduisent les pressions ou tensions exercées sur les surfaces s, s′, . . . Donc les sommes des projections algébriques de toutes ces forces et de leurs moments linéaires sur les axes des x, y, z devront se réduire à zéro.* [4, p 44]
>
> Let ξ, η, ζ be the centre points of the surface *s* (one facet of the surface of a body with finite mass *M*). Let *m* be an infinitely small mass (mass point) contained in the volume v. Let φ be the acceleration acting on this mass if the solid body is in equilibrium. Let *X*, *Y*, *Z* be the Cartesian components of the acceleration φ, and ξ_0, η_0, ζ_0 the coordinates of the centre of gravity of the mass *m*. If φ is the same throughout *m*, equilibrium must exist between the driving force $m\varphi$ acting on the point (ξ_0, η_0, ζ_0) and the forces to which the pressures and tensions acting on the surfaces s, s', . . . reduce. Thus the directional components of all these forces and their linear moments on the coordinates *x*, *y*, *z* must sum to zero.

Neither textbooks nor personal discussions ever raised, or even permitted, the question of why the principles of Newton's mechanics should be irrelevant in continuum mechanics. Forces are vectors that act upon a point. Newton considered their interaction with a discrete solid body. He found that a force **f** acting on a point *P* on the surface of the body causes a linear displacement on the body if **f** is collinear with the radius **r**, where **r** is the position vector of *P* with respect to the centre of mass *Q* of the body, and a force perpendicular to **r** produces a torque (figure 1(a)). The rotational equilibrium condition is $\int \mathbf{f} \times \mathbf{r}\, \mathrm{d}A = 0$ or, if **f** and **r** are defined as functions of the orientation θ, $\int \mathbf{f} \times \mathbf{r}\, \mathrm{d}\theta = 0$. Here **r** is a lever. Hence, the vectors **f** and **r** interact with one another; the shape of the body as defined by $\mathbf{r}(\theta)$—that is, the spatial configuration of surface points—represents half the data set required for the equilibrium condition, the other half is the spatial configuration of **f**. That is, the equilibrium condition has two degrees of freedom. However, the orientation of the surface *A* is entirely irrelevant. Note that both $\mathbf{f} \cdot \mathbf{r}$ and $|\mathbf{f} \times \mathbf{r}|$ are Joule terms, i.e. they refer to work, in this case acceleration work.

Since Euler, however, forces in continuum mechanics are believed to interact with planes. A normal force is normal to a plane in space; a shear force is thus parallel to a plane. Consequently, the driving agent in continuum mechanics is taken to be a form of pressure $|\mathbf{f}|/A$. In citation 1, **r** is the distance ($\xi-\xi_0$, $\eta-\eta_0$, $\zeta-\zeta_0$). This distance is understood by Cauchy as the topological distance of the point of interest (in a plane) relative to a coordinate origin

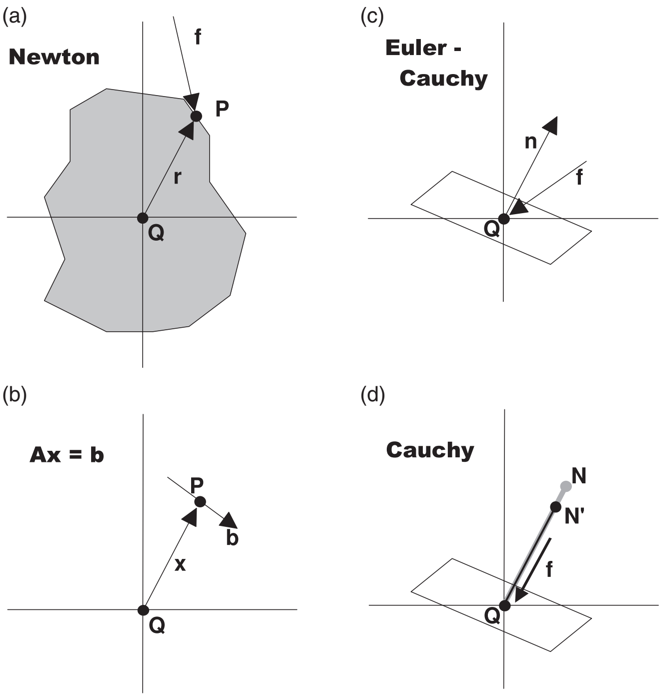

Figure 1. Vector relations in the Euclidean space. (a) Newton's definition of a torque relates a force **f** and a radius vector **r**. The point of action P of **f** and the centre of mass Q are different points. (b) Properties of vector spaces. The linear equation $\mathbf{Ax} = \mathbf{b}$ assigns a vector **b** to the point P indicated by **x** relative to Q. P and Q are different points. A plane in space indicated by the Hesse notation would be at P and perpendicular to **x**. Planes containing Q cannot be indicated. (c) Euler's convention of stress has a force **f** acting on a point Q in a plane whose orientation is given by **n**. **n** can only indicate a plane at Q, but nowhere else. (d) Cauchy's stress vector **f** acts on the point of action Q in the plane. The distance QN (grey) is then thought to contract to QN' (black).

only—which is arbitrary—but not as a lever in the sense of Newton. Cauchy thereby deprived himself of a degree of freedom which is offered by nature. The point (ξ_0, η_0, ζ_0) is mentioned on the following page in a cross product equation which is perfunctorily set to be zero, and never again. The concept of the lever has been abolished in continuum mechanics. I have yet to find a physical rationale as to why the deviation from Newton's principles is justified. There is no reason, other than Euler's contention, which is unsubstantiated.

Understanding of the nature of solids and fluids

> Citation 2: *Si le corps que l'on considère se réduisait à une masse fluide, il y aurait, en chaque point, égalité de pression en tout sens, et chaque pression serait perpendiculaire au plan qui la supporterait. Alors les pressions exercées en un point quelconque, et du côté des coordonnées positives, contre trois plans perpendiculaires aux axes des x, y, z, seraient dirigées parallèlement à ces axes, mais dans le sens des coordonnées négatives.* [6, p 111]
>
> If the body under consideration were made of a fluid, we would have similar pressure in all directions at every point, and every pressure were perpendicular to the plane on

which it would act. Then the pressures exerted on an arbitrary point, and along the positive coordinate directions, against three planes perpendicular to the axes x, y, z, would be oriented parallel to these axes, but in the sense of the negative coordinates.

The properties of solids and fluids are outlined in more detail in [4] and [8], but the result is the same: Cauchy's approach is macroscopic. To some degree, the lever has not been missed because his understanding of a solid is rudimentary in a modern light. The basic difference between a solid and a fluid is, in Cauchy's view [3, 4], that solids can maintain shear forces whereas fluids cannot, hence only normal forces act in fluids. Modern textbooks on fluid dynamics no longer subscribe to this view; the stress tensor is commonly explained in all nine components even for fluids. But the point remains that there is no real sense of material coherence in Cauchy's theory, as if the concept itself has not been invented yet. There is indeed no physical term or concept in his theory that could be interpreted to indicate a material strength, other than a proportionality constant (the spring constant). This is not enough; such a view is entirely innocent—literally—of bonds: permanent bonds in solids, transient bonds in fluids, of the atomic nature of matter, and of diffusion.

Bonds are forces which must be included in the equilibrium conditions, or else the latter are not complete. The existence of bonds is macroscopically felt as hardness. A permanent static tension can only exist in a bonded continuum. It is the continuity of bonds, and not merely the continuity of mass distribution, that distinguishes a distance in a bonded continuum of mass from a distance in a gas, only the former can be a lever. In a fluid, it is subject to decay due to diffusion. A consideration of mass distributions without any mention of bonds ignores the difference between a solid and a dense gas in favour of the latter. A theory of stress in solids that does not take bonds into account cannot possibly be correct. The existence of bonds in materials is indicated by the condition that the internal pressure $(\partial U/\partial V)_T \neq 0$; but that had to await the advent of thermodynamics.

Cauchy had no conception of physical work [5, 9, 14]. He must have realized that all displacements must cancel for an isochoric deformation, but he did not see that if the paths cancel, the Newtonian work cancels. He possibly believed the zero result to be correct because then $E_{\mathrm{kin}} + E_{\mathrm{pot}} = \mathrm{const}$ would be observed (cf citation 5, $\kappa = 0$; [17, 19]), the only energy conservation law known to him. Cauchy never mentions the term *travail* anywhere [1–15]. This is not surprising, it was not known yet; Coriolis would discover the concept of work only 2 years later in 1829. Beyond that, elastic deformation work is akin to *PdV*-work which was defined by Joule in 1845. Proper understanding of the difference between Newtonian *work done in a system* through a conservative process and thermodynamic *work done upon a system* through a non-conservative process requires an understanding that was at best tentatively available before 1870, the year in which the term and concept of a *state function* was coined in the correspondence between Clausius, Joule and Gibbs. Cauchy was a mathematician fascinated by ellipsoids who perceived deformation as a geometric problem in the context of Newtonian mechanics only. But work and energetic considerations are the key to physics (cf citation 5).

Vector spaces and field theory

Citation 3: *Concevons d'abord que le volume v prenne la forme d'un prisme droit, dont les deux bases soient représentées par s et par s′. On aura s′ = s; et, si, les dimensions de chaque base étant considérées comme infiniment petites du premier ordre, la hauteur du prisme devient une quantité infiniment petite d'un ordre supérieur au premier, alors, en négligeant,* [. . .] *les infiniment petits d'un ordre supérieur au second, l'on trouvera (p cos λ + p′ cos λ′)s = 0* [. . .], *et l'en conclura p′ = p,* *cos* $\lambda' = -$ *cos* λ [. . .]. *Ces dernières équations ont rigoureusement lieu dans le cas où la hauteur du prisme s'évanouit, et comprennent un théorème dont voici l'énoncé.*

1[er] Théorème. *Les pressions ou tensions exercées, en un point donné d'un corps solide contre les deux faces d'un plan quelconque mené par ce point, sont des forces égales et directement opposées.* [4, p 46]

First of all, let us imagine a volume v to take the shape of a rectangular prism, with basal and top planes s and s'. Thus $s' = s$; if the dimensions of each plane are considered as an infinitesimally small quantity of first order (in a polynomial series), the height of the prism becomes an infinitesimal quantity of higher rank. Thus if the higher order terms are ignored, we obtain $(p \cos \lambda + p' \cos \lambda')s = 0$; so it is concluded that $p' = p$, $\cos \lambda' = \cos \lambda$. These last equations are absolutely valid in the case if the height of the prism vanishes, and can be condensed into a theorem.

1st theorem. The pressures and tensions exerted at a given point within a solid upon the two sides of a plane passing through this point, are equal in magnitude and opposite in direction.

This theorem is known as the Cauchy lemma, often expressed as

$$\mathbf{f}_{-x} = -\mathbf{f}_x. \tag{1}$$

From a modern point of view, there are some physical and some mathematical conflicts in the argument. The physical ones are discussed following citation 5; the mathematical inconsistencies are [16, 17, 19]: two vectors, **f** and –**f**, are assigned to the same point x; and: the plane s has two notations, x and $-x$, depending on which side is considered.

The relation of forces and planes on which they act is such that a force acts on every point of s. A modern field theory is given by

$$\mathbf{A}x = \mathbf{b}, \tag{2}$$

by which a vector **b** is assigned to any point P indicated by **x** relative to a point of interest Q of which the field property tensor $\mathbf{A}(Q)$ is a function of location (figure 1(b)). This operation complies with the minimum requirements for vector spaces which ensure that no two objects (vectors, points or planes) can be assigned the same notation, that no two notations can indicate the same object and that no object is without notation, such that minimum arithmetical logic is preserved.

Continuum mechanics clearly calls for a field theory. But in equation (2), the Hesse notation for objects in space is used. One of its properties is—and must be for any logical vector space—that the zero object $\mathbf{0}$ exists such that **x** and –**x** are two different objects, and $\mathbf{x} - \mathbf{x} = \mathbf{0}$. Equation (1) cannot be reconciled with these requirements. The zero object $\mathbf{0}$ does not exist in the convention employed by Euler and Cauchy. A plane with notation $\mathbf{n} = [1 \quad 0]$ is at the point $[1 \quad 0]$ in the Hesse notation, and oriented perpendicular to $[1 \quad 0]$. In Cauchy's convention, the plane $[1 \quad 0]$ is at $[0 \quad 0]$, and oriented perpendicular to $[1 \quad 0]$. The notation $\mathbf{n} = [-1 \quad 0]$ delivers the same result. Points other than Q cannot be described, and the operation $\mathbf{n} - \mathbf{n}$ is meaningless; the notation used in the Euler–Cauchy theory does not relate to the Euclidean space. The point $[1 \quad 0]$ is itself meaningless, physically and geometrically; it is a direction indicator standing for a ray, but not for a distance in space; it could be any point $[x \quad 0]$ chosen by convention. Hence, continuum mechanics does not have a spatial metric.

This needs to be seen in historical context. Standard concepts that go without saying today had yet to be established then. Hesse (1811–1874) became scientifically active only

10 years later; the full set of rules for vector spaces and tensor algebra were worked out by Grassmann (1809–1877) in 1862; and the physical problem that made the scientific world appreciate field theories was the research on electromagnetics by Faraday (1791–1867) in the 1860s, after Cauchy's death in 1857. (Note the definition of a plane by points O, O', O'' in citation 6, which would be unacceptable today where Hesse's and Grassmann's systematics are universally accepted. The plane in question contains the coordinate origin.)

Reading Cauchy is a somewhat disconcerting experience (to this author at least) because he often switches from one reference point to the next within a few sentences. Commonly, he refers to planes that contain his chosen origin, but the Hesse notation cannot assign a notation to them because $\mathbf{x}$ is then a zero vector, and $P = Q$ (figure 1(c)). Thus if Cauchy describes the facets of the surface of a volume as s, s', s'', $s'''\dots$ [4, p 43] he is clearly unaware that there might be a major problem; however, some planes can be notated, some cannot. Cauchy evidently realized that at least a minor problem existed here which he solved consistently—in his view, but not in the light of later insight—by restricting his considerations to the positive coordinate directions only, apparently assuming that somehow things would sort themselves out on the other side. Cauchy did not realize that he often used two sign conventions simultaneously. Surely he understood *pression* and *tension* as opposites, ditto for *condensation* (shortening) and *dilatation* (stretch). In the positive quarter of a Cartesian coordinate set, a *tension* and *pression* may have positive and negative sign respectively, but in the negative quarter, the Cartesian sign convention is in conflict with the physical contrast of tension and compression. Thus he often implies a sign convention that distinguishes inward and outward, such that a compression would always have negative direction and negative sign on both sides of the origin; but he did not say so. He did not think of a system he could separate from a surrounding. He always explicitly referred to Newton's third law as an equilibrium condition. He was clearly unaware of the thermodynamic equilibrium condition, he had no inside and outside; his reference object was a point, the origin Q.

Generality

Citation 4: *Soient maintenant p', p'', p''' les pression ou tension exercées au point (x, y, z) et du côté des coordonnées positives* [. . .] *Enfin concevons que le volume v, prenant la forme d'un parallélépipède rectangle, soit renfermé entre les trois plans menés par le point (x, y, z), et trois plans parallèles menés par un point très-voisin $(x + \Delta x, y + \Delta y, z + \Delta z)$. Les pressions ou tensions, supportées par les faces du parallélépipède qui aboutiront à ce dernier point, seront à très-peu près $p'\Delta y\Delta z$, $p''\Delta z\Delta x$, $p'''\Delta x\Delta y$.* [. . .] *Quant aux pressions ou tensions supportées par les faces qui aboutissent au point (x, y, z), elles seront, en vertu du 1er théorème, respectivement égales, mais directement opposées à celles qui agissent sur les faces parallèles aboutissant au point $(x + \Delta x, y + \Delta y, z + \Delta z)$.* [. . .] *Ajoutons que les centres de gravité des six faces du parallélépipède se confondront avec leurs centres de figure, et seront situés sur trois droites menées parallèlement aux axes des x, y, z par le centre du parallélépipède, c'est à dire, par le point qui a pour coordonnées $(x + \frac{1}{2}\Delta x, y + \frac{1}{2}\Delta y, z + \frac{1}{2}\Delta z)$.* [4, pp 46–47]

If p', p'', p''' are the pressures and tensions acting on the point (x, y, z) along the positive coordinates. Let us also assume that the volume v has the shape of a rectangular cuboid confined by the three planes (containing the coordinates) through the point (x, y, z) and three planes parallel to them through the point $(x + \Delta x, y + \Delta y, z + \Delta z)$ close by. The pressures and tensions acting on the faces of the cuboid running through the latter point are very similar to $p'\Delta y\Delta z$, $p''\Delta z\Delta x$, $p'''\Delta x\Delta y$. Due to theorem 1 (cf citation 3), the pressures and tensions acting on the faces running through the point (x, y, z) are identical (in magnitude), but opposite in direction to those that act on the planes through the point $(x + \Delta x, y + \Delta y, z + \Delta z)$. We add that the centres of gravity of the six faces of the cuboid are on top of one another and the centre of the cuboid, i.e. the point with coordinates $(x + \frac{1}{2}\,\Delta x, y + \frac{1}{2}\,\Delta y, z + \frac{1}{2}\,\Delta z)$.

Cauchy then goes on—perfunctorily again—to claim that the torque is balanced, and that the state of stress is therefore orthogonal by nature.

It is perfectly clear from this citation that Cauchy saw that the volume element has a specific shape, that of a cube. The Δx etc can safely be assumed to be of similar length; they cannot be arbitrary (cf citation 1). It is therefore clear that Cauchy's conclusion that the state of stress is orthogonal is not a general solution at all, but a function of the chosen volume shape. Maybe it did not matter to him because he let the volume vanish anyway (citation 3) which, however, is in conflict with the potential theory (citation 5).

(Some guessing may be permitted here to do justice to Cauchy. He did not attempt an analysis of boundary conditions. The dependence of the rotational equilibrium on body shape in Newtonian mechanics must have been known to him, but since he mainly thought in terms of planes, following Euler, not of bodies, it is possible that his mathematical mind seduced him: the observation that in the orthogonal loading state the plane normal vector **n** and the principal axes of stress are collinear, may have been irresistibly attractive. The bait was very close to what Grassmann would name an eigendirection, 35 years later—if **n** were physically or mathematically relevant, which is not the case in Newton's mechanics (citation 1). For a vector field given by equation (2), an eigendirection is any direction for which **x** and **b** are collinear. It was Cauchy's mistake to follow Euler (1707–83), whose immense authority one could hardly escape in the early 19th century.)

If it is assumed that Cauchy envisioned something like a modern field theory—a function like equation (2) that assigns a vector to any point in space, such that all forces at all points acting on the surface of his volume element are known—he is still correct in assuming that the forces on opposite sides are similar in magnitude, but opposite in direction; but this implies only point symmetry, not orthogonality. By choosing a cube as a body shape, he preconditioned the state of loading to be orthogonal, because a cube can only be in equilibrium with a force field that has at least orthogonal properties. That is, the shape of v acts as a hidden boundary condition. Other shapes offer more freedom. For example, a volume element of elliptical shape with half-axes $x_1 = a$, $x_2 = {}^1/_a$ is described by a radial vector field of the form

$$\mathbf{r} = \begin{bmatrix} a\ \cos\theta & \frac{1}{a}\sin\theta \end{bmatrix} \tag{3}$$

where θ is the angle of the direction from $x_1 = 0°$. The ellipse is in equilibrium with a vector field **b** (cf equation (2)) for which

$$\mathbf{A} = \begin{bmatrix} 0 & a \\ a^{-1} & 0 \end{bmatrix}. \tag{4}$$

The eigendirections of the field **b** are

$$\begin{bmatrix} 1 \\ a^{-1} \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -a^{-1} \end{bmatrix}, \tag{5}$$

i.e. they are non-orthogonal, the field **b** is monoclinic (figure 2). Cauchy's analysis of boundary conditions is insufficient. The orthogonality of stress is, and has never been more than, a premature contention.

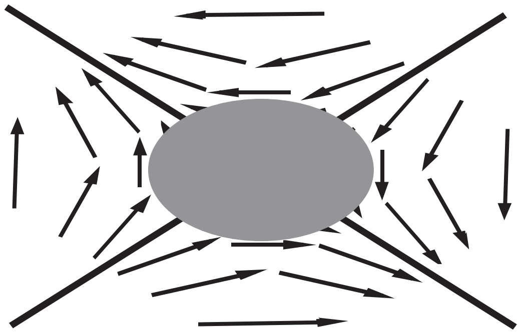

Figure 2. Monoclinic vector field with non-orthogonal eigendirections. A body subjected to this force field is in equilibrium if the ellipticity of force field and body shape are mutually reciprocal. A sphere or cube can only be in equilibrium with a vector field with orthogonal or higher symmetry.

Potential theory

Citation 5: *Soit M la masse d'un corps solide en équilibre, m une particule ou portion infiniment petite prise au hasard dans cette masse, x, y, z les coordonnées de la particule m* [. . .], *et ρ la densité du corps solide au point (x, y, z). Si l'on nomme p′, p″, p‴ les pressions ou tensions exercées au point (x, y, z) et du côté des coordonnées positives,* [. . .]

De plus, si, après avoir fait passer par le point (x, y, z) un plan quelconque, on porte, à partir de ce point, et sur chacun des demi-axes perpendiculaires au plan, deux longueurs équivalentes, la première à l'unité divisée par la pression ou tension exercée contre ce plan, la seconde à l'unité divisée par la racine carrée de cette force projetée sur l'un des demi-axes que l'on considère, ces deux longueurs seront les rayons vecteurs de deux ellipsoïdes dont les axes seront dirigés suivant les mêmes droites. A ces axes correspondront les pressions ou tensions principales dont chacune sera normale au plan qui la supportera, et parmi lesquelles on rencontrera toujours la pression ou tension maximum, ainsi que la pression ou tension minimum. [9, pp 160–161]

Let M be the mass of a solid body in equilibrium, m be a particle or infinitely small part at some point in the mass, x, y, z be the coordinates of the particle m, and ρ be the density of the solid body at the point (x, y, z). If p', p'', p''' are the pressures and tensions exerted at the point (x, y, z) along the positive coordinates, [. . .]

Furthermore, if we let some plane pass through the point (x, y, z), and then draw two equivalent lengths from this point along both the traces of the coordinates on this plane—the first one: the reciprocal of the pressure or tension exerted on this plane, the second one: the reciprocal of the square root of this force projected on one of the half-axes considered—these two distances are the radius vectors of two ellipsoids whose axes are mutually perpendicular. These axes represent the principal pressures and tensions each of which is normal to the surface on which it acts, and among which the maximum pressure and minimum tension are always found.

Si l'on désigne par ε une quantité positive ou négative qui représente la dilatation ou la condensation linéaire du corps solide mesurée au point (x, y, z) sur une droite *tracée de manière à former avec les demi-axes des coordonnées positives les angles* α, β, γ, *on aura* [. . .] $(1+\varepsilon)^{-2} =$ [. . .]

Il en résulte que, si, à partir du point (x, y, z) on porte sur les droites en question une longueur équivalente à $1 + \varepsilon$, *l'extrémité de cette longueur sera située sur la surface d'un ellipsoïde dont la construction indiquera les rapports constants entre les dilatations ou condensations linéaires mesurées dans les diverses directions autour du point (x, y, z). Les dilatations ou condensations correspondantes aux trois axes de l'ellipsoïde sont celles que nous avons nommées principales. Les autres se trouvent symétriquement distribuées autour des mêmes axes. Ajoutons que, si l'on désigne par* ε', ε'', ε''' *et v des quantités positives ou négatives, propres à représenter (1) les dilatations ou condensations principales, (2) la dilatation ou la condensation du volume au point (x, y, z), on aura* $1+ v = (1 + \varepsilon')(1 + \varepsilon'')(1 + \varepsilon''')$. [9, pp 163–164]

If ε is a positive or negative quantity representing the linear stretch or shortening of the solid body at the point (x, y, z) on a line that forms with the positive coordinates the angles α, β, γ, we obtain $(1 + \varepsilon)^{-2} = \ldots$

If lines are drawn from the point (x, y, z) along the lines in the question with lengths $1 + \varepsilon$, the outer point of this distance term marks the surface of an ellipsoid whose construction indicates the constant relations among the linear stretches and shortenings in the various directions around the point (x, y, z). The principal stretches and shortenings are those along the three main axes of the ellipsoid. The others are distributed symmetrically about these axes. If ε', ε'', ε''' and v are positive or negative quantities representing (1) the principal stretches and shortenings, (2) the positive or negative volume change at (x, y, z), we obtain $1 + v = (1 + \varepsilon')(1 + \varepsilon'')(1 + \varepsilon''')$.

In summary: the force (understood to be force per area) acts at the point of interest $(x, y, z) = Q$ on the plane that passes through it. The effect of this force acting on Q is that a point N some distance away—say, the other end of Hooke's spring—and whose location in space cannot be described in this notation, is displaced from N to N', such that the spring contracts (figure 1(d)). It is hard to see why the spring should do this.

Conventional continuum mechanics surely offers a sense of direction, but not of space. Neighbouring points must always be supplied by auxiliary means. This deficiency led to the development of the finite element method, which a proper field theory would not need. The Fourier series method works much more simply, and it complies with the requirements for vector spaces; but it needs a base distance to build up upon—which was offered by the length l_0 of Hooke's spring, after all, and by Newton's radius vector $\mathbf{r}$ (figure 1(a)). The existence of this distance term in real space is required in the potential theory where it is called the zero potential distance. It may be infinite or finite; if it is finite, it is commonly set to have unit length, but it cannot be zero. In thermodynamics, it may be identified as the radius of the thermodynamic system in the standard state. The zero potential distance is required to define work, because a point cannot be displaced with respect to itself. Cauchy introduced this unit distance in his strain theory [5, 9], but he let it vanish identically in his stress theory (the distance s-s' in citation 3), and with it any sense of space. It is therefore incompatible with Newton's mechanics, potential theory, and thermodynamics [17, 19]; in fact, since his theory of strain does contain the metric whereas his theory of stress does not, Cauchy's two theories are incompatible with one another.

But the conflict of Cauchy's theory with reality is deeper. The potential theory distinguishes two fundamentally different classes of physical processes: those for which the energy of a system is constant ($\partial^2 U/\partial x^2 = 0$, Laplace condition), and those that cause the energy of a system to vary ($\partial^2 U/\partial x^2 = \kappa$, Poisson condition). The former can be correlated with Newtonian mechanics and the conservative energy conservation law $E_{\text{kin}} + E_{\text{pot}} = \text{const}$; the Laplace condition is one way of saying that the kinetic system is isolated, or that the thermodynamic system is in the standard state (the zero point by convention). The Poisson condition is the basis for continuum physics, the physics of changes of state, and the first law $\mathrm{d}U = \mathrm{d}w + \mathrm{d}q$ since κ can be interpreted as a measure of the work done upon a system. Judging by its entire mathematical and physical structure, Cauchy's theory is conservative and unsuited to describe a change of state [19].

The volume of the prism in citation 3 is thought to vanish identically. The prism is a system of mass subjected to external loading. Hence, the forces acting on s and s' are external forces, exerted by a surrounding upon a system. Since equilibrium is assumed, the system must counter the external forces by means of internal forces, the compressibility or material strength. They are not mentioned—not here, and nowhere else in any of the papers considered for this review [1–15]. Because they were never considered, elasticity and stress appear to be akin to a conservative process because κ is then zero by default, which is a gross mischaracterization in modern understanding. It is, however, entirely in harmony with the state of physics in the 1820s, 20 years before the discovery of the first law of thermodynamics.

Cauchy assumed that the lateral faces of the prism and forces acting on them are 'terms of higher order', i.e. they are believed to vanish faster than the first-order terms acting on s and s' as the distance s–s' approaches zero. This is purely a conjecture, and it cannot be right, as becomes clear if the prism volume is thought to approach zero isotropically, such that its shape is preserved as $V \to 0$: if s-$s' = r$, all facets are $\propto r^2$ and would vanish at similar rate. In citation 3, the prism changes shape as s-$s' \to 0$ which would render the equilibrium conditions arbitrary; this is impossible. The planes s and s' are considered free planes, but this they are not; s plus s' plus the four lateral faces form a *closed surface* enclosing a mass m, and if $V \to 0$, mass $\to 0$; but mass and charge φ are always proportional in a given state.

Cauchy took the scale independence of $P = |\mathbf{f}|/A$ in Newton's definition of pressure for granted. However, this is correct for free planes only. On closed planes in a continuum of mass, the source density κ is scale independent; since $\varphi \propto$ mass $\propto V$, it follows from

$$\int \mathbf{f} \cdot \mathbf{n}\, \mathrm{d}A = \int \nabla \cdot \mathbf{f} \mathrm{d}V = \int \kappa \mathrm{d}V = \varphi \tag{6}$$

that $|\mathbf{f}|/A \to \infty$ as $V \to 0$ [16, 19]. The limit does not exist. It follows that Cauchy's tensor does not exist as a mathematical term. In contrast, the thermodynamic pressure $P = \partial U/\partial V$ remains scale independent always as $V \to 0$, which ensures its universality. This argument alone is *necessary and sufficient* to prove that Cauchy's stress tensor cannot exist [19].

The operation by which the volume is reduced to a point (s-$s' \to 0$, citation 3) is known as the continuity approach. It is supposed to be the transition from a discrete body to the continuum of points. The idea behind this is the concept of Newtonian point mechanics. But the conceptual context of point mechanics does not apply here. A discrete body with finite volume V and finite mass content in freespace can be reduced to a point if a surface A enveloping it does not run through mass. The reference mass in the system must remain invariant as V vanishes. V can then be ignored, but the mass is thought to be concentrated in a point and still finite. It can then be considered a *point source* of body forces, which is mathematically much simpler to handle. But obviously this does not apply to continuous mass distributions, bonded or not. A continuous mass distribution such as the interior of a solid must not be reduced to a point since the reference mass must then vary with V; it is therefore a *distributed source*. The classical example is the thermodynamic system where n and V can be any fixed number in $PV = nRT$ except zero; they are then by convention set to 1 in the standard state. Cauchy's continuity approach is not permitted by potential theory.

The shape of V in equation (6) is arbitrary only if A does not run through mass, that is, if A marks the boundary of a volume V containing a discrete body in freespace. The condition does not apply to the interior of a solid. The divergence theorem can only consider radial fluxes. Therefore it always applies to body forces, it may apply to heat flow; but it applies to mechanical forces only if all forces acting on the system V are normal forces and radius-parallel, i.e. for the specific boundary conditions of a hydrostatic pressure change. If radius-normal forces can do work upon V, the shape matters (citation 1), constraints must be found for it, and φ is then *not* a measure of the total work done upon the system, but at best an incomplete answer. In an anisotropic elastic deformation, clearly both normal and shear forces do work; but whereas at least the effect of a normal force is easy to understand—a shortening or a stretch of a radius vector—it is not at all clear what the effect of a shear force could be, and how a shear force does physical work. But within the frame of Cauchy's theory it is plainly impossible to pose the question (one reason being simply the impossibility to define shear in a geometric system that does not relate to the Euclidean space, cf citation 3). The effect cannot be a free rotation as in Newtonian mechanics because the solid is internally bonded (citation 2), and rotational equilibrium exists by definition. Whereas the work done by normal forces is ill-understood—the perpetuum mobile condition $\kappa = 0$ cannot be the right answer—the contribution by shear forces to the total elastic deformation work has been overlooked entirely, conceptually and numerically, in theories based on Cauchy's which commonly concentrate on tensor invariants; but cross product contributions must be found through an integral over the surface of a volume, which Cauchy's theory does not have.

This author has not seen a published rationale to clarify whether strain ε (defined by Lagrange in 1787) is a state function (a concept 80 years younger) or not, let alone rigorous proof that it is one. The work term $\sigma\,\mathrm{d}\varepsilon$ today in use is the product of two terms of which the traces σ_{ii} and ε_{ii} are both zero for an isochoric deformation; hence their product must be energetically empty [19]. The stress tensor σ does not even exist (see above) [16, 19]. Strain ε is a valid geometric term, but it was never meant to be a measure of work, because work was not known to Lagrange; it lacks the mathematical properties to serve as one precisely because its trace can be zero, and experiments show that it is indeed not a state function: elastic work varies with boundary conditions for identical strain, suggesting that the proper term to measure deformation work is not strain, but displacement [17], which is known to contain more information anyway. Taking ε as a state function has never been more than an assertion dating from the mid-19th century. Deformation work cannot be defined in Cauchy's theory because of its energetically conservative character.

Thermodynamics and the first law

> Citation 6: (The molecule of interest m_0 with coordinates (a, b, c) is a point in a plane defined by three points O, O', O''. The surface element s is a small unit area within the plane $OO'O''$ with the molecule of interest at its centre. Molecules m and pressures and tensions p on the positive side of $OO'O''$ are indicated by primes, those on the negative side by subscripts.) *Les actions exercées par les molécules* $m_1, m_2, \ldots$ *sur les molécules* $m, m', m'', \ldots$ *sont égales et directement opposées aux réactions exercées par les dernières sur les premières; et il est clair qu'on n'altère pas sensiblement la résultante des ces actions ou de ces réactions, lorsqu'aux molécules* $m, m', m'', \ldots$ *ou* $m_1, m_2, m_3, \ldots$ *on joint celles que se trouvent précisément situées dans le plan* $OO'O''$*. Cela posé, le pressions ou tensions* p*'s,* $p_1 s$ *supportées* [...] *par la surface*

> *élémentaire s, pourront être considérées come deux forces égales, mais directement opposées, et l'on devra en dire autant des pressions* p', p_1, *exercées au point* (a, b, c) *contre les deux faces du plan* $OO'O''$. [11, p 217]
>
> The effects exerted by the molecules on the negative side upon the molecules on the positive side of the reference plane are equal in magnitude and opposite in direction to the effects by the latter upon the former; and it is clear that the total effect of these actions and reactions is not changed sensibly, if the molecules within the reference plane are considered. Thus the pressures or tensions p*'s*, $p_1 s$ exerted upon the surface element s can be considered as two forces equal in magnitude and opposite in direction, and one can just as well call them pressures p', p_1 exerted upon the point (a, b, c) from either side of the reference plane.

In [10, 11], Cauchy's views of a solid are amazingly modern in comparison to [4]. The atomic nature of matter plus the interaction of atoms by means of electromagnetic forces was in his days at best a vague hypothesis. As in citation 1, Cauchy considers forces acting on planes, not on volumes that might be interpreted as a thermodynamic system, and which can react to loading by deforming. His 'molecules' are—in a modern view—discrete bodies in freespace. They are not bonded; it is clear that Cauchy did not understand the distance from one molecule to the next as a fixed equilibrium distance, stable by nature, and subject to change only if external work is done upon it. If it were so, the cause of the effect done on the molecule in the reference plane must be beyond the molecules nearby, but it cannot be caused by the nearby molecules themselves. It would be necessary to distinguish a discrete system with finite spatial extent upon which a surrounding can do work, and the smallest system possible must consist of at least two atoms to account for the energy stored in the bond between them (cf [18, p 2658]). Such a bond length would offer itself as a zero potential distance (citation 3). Thus, since the thermodynamic system is finite and not a point, Cauchy's theory is at variance with the thermodynamic theory.

This should not be surprising, since the first law became known only 20 years after [1–15] were published. Only then could conservative processes (observing the Laplace condition div $\mathbf{v} = 0$) and non-conservative processes (following the Poisson condition div $\mathbf{v} = \kappa$) be discerned, and conservative work and non-conservative work could be—or should have been—understood as profoundly different in nature: Newtonian work is done *within* a system, such that $E_{\mathrm{kin}} + E_{\mathrm{pot}} = \mathrm{const}$; PdV-work is a work done *upon* a system, such that $U_0 \rightarrow U_1$. But this rather essential distinction is seriously blurred in continuum mechanics to this day [19, p 4868ff]; the first law of thermodynamics has in fact never arrived in its conceptual content in continuum mechanics, only by name.

Thermodynamics requires the distinction of system and surrounding, e.g. in the equilibrium condition $P_{\mathrm{syst}} = P_{\mathrm{surr}}$. It therefore considers forces from two different origins: one source of forces is within, the other outside the system. It does not know an equation of motion, it cannot: neither does the inertial mass m (kg) have a place in a thermodynamic context, nor is there a time term in the equation of state. Instead, the thermodynamic mass is measured in mol, the number of atoms as a measure of their element-specific electromagnetic properties, and a thermodynamic force is given by $\mathbf{f} = \mathbf{e}_i\, \partial U/\partial x_i$, which is a force field derived from a potential, but not a free vector like $\mathbf{f} = m\mathbf{a}$. The latter refers implicitly to the motion of a discrete body of inertial mass in freespace. Inside a solid it is out of place.

The traces of conservative—pre-thermodynamic—concepts can be found throughout continuum mechanics. It starts with an equation of motion, all the way to the material derivative in plasticity. It uses ρ, the density of the inertial mass, as density term, instead of mol V^{-1}. Both the use of time and inertial mass are indirect references to $\mathbf{f} = m\mathbf{a}$, and thus to concepts which have no place in thermodynamics which is concerned with the physicochemical properties of matter, i.e. the electromagnetic properties of n atoms, bonded or not. Elastic deformation is by nature a change of state in the sense of the first law of thermodynamics such that an elastic potential builds up upon loading. The simplest elastic deformation is thus the volume change of an ideal gas due to an isotropic pressure increase. Therefore, standard thermodynamics and the first law must be the starting point for a theory of elasticity applying to denser materials and anisotropic boundary conditions [18]. Without this insight, and without observing it in thought and concept, it is not possible to understand elasticity properly.

Real materials

The incompatibility of Cauchy's theory with general vector space systematics, especially the vanished zero potential distance, gave it a mathematical structure which sets it apart from all other physical theories involving vector fields. An auxiliary construction, the finite element method, is required to link it to the Euclidean space.

The Cauchy theory survived because its predictions are often close to reality because of a coincidence. Due to the principle of least work, an elastic deformation will always assume the state of deformation with the highest symmetry properties possible permitted by the boundary conditions, i.e. the spatial gradients are minimized, and forcing the material into a deformation state of lesser symmetry costs extra work [18]. Therefore, the symmetry state of a deformation commonly complies with the properties assigned to Cauchy's tensor which is at least orthogonal. The critical deformation state is therefore the one which has lesser symmetry than Cauchy's stress, i.e. monoclinic simple shear. The current theory of deformation and flow fails systematically for simple shear in the elastic, viscous and plastic field. These gaps in the current understanding are well known. To name just a few,

- in the elastic field, it fails to predict the Poynting effect, that an isotropic solid subjected to elastic simple shear will dilate, anisotropic materials will always dilate;
- in the viscous field, it fails to predict the generation of turbulent flow;
- in the plastic field, it fails to predict the experimentally observed fact that plastic simple shear costs substantially less work per chosen unit strain ε than plastic pure shear.

Cauchy's theory does not consider the existence of bonds in continuous media; hence it cannot predict effects that come about by breaking bonds at the reversible–irreversible transition. A modern field theory that takes the existence of bonds into account predicts all these phenomena readily [18]. It has been found that this transition triggers a bifurcation by which the loaded state will relax into one of two possible, energetically equivalent states with opposite handedness. This bifurcation offers a straightforward mechanism for the generation and orientation of cracks at the elastic-brittle transition, for the generation and the geometric properties of the initiation of turbulence in viscous flow, and of certain instabilities in plastic flow leading to the generation of sheath folds.

Historical notes

In his essay on elasticity Maxwell [21] wrote, 'There are few parts of mechanics in which theory has differed more from experiment than in the theory of elasticity'. He dismissed purely mathematical reasoning to the extent that he did not even mention Cauchy, giving people with known experimental experience—Navier, Poisson, Lamé—much more credit. Maxwell was the first to this author's knowledge to clearly postulate the existence of bonds in solids:

'Solid bodies are composed of distinct molecules which are kept at a certain distance from each other by the opposing principles of attraction and heat'. This distance term makes his understanding superior to that of Cauchy. Even Maxwell did not mention the thermodynamic equilibrium condition yet; this is just to illustrate how foreign it was to the workers of the time. His paper shows that bonds were still very little understood in the mid-19th century. It confused workers that a solid can show general elasticity whereas a fluid can react elastically to isotropic loading, yet yields readily to anisotropic loading. Today, we know that elastic-reversible loading and irreversible dissipative relaxation are two pairs of shoes. The latter requires understanding of van der Waals forces and the second law, which was still beyond the horizon in 1850. The key is not in mechanics, but in thermodynamics: pressure $P = (\mathrm{d}U/\mathrm{d}V)_S$; a hydrostatic pressure increase is isentropic, whereas anisotropic loading reduces entropy. An anisotropically loaded material is therefore potentially out of equilibrium with itself. A solid with permanent bonds may sustain this, a fluid does not.

An approach to elasticity which is entirely independent of Cauchy was outlined by Helmholtz. He did not use an equation of motion, but derived a vector field from a potential energy term, and he separated a system of unit size from the surrounding, as in thermodynamics proper, keeping the zero potential distance intact. Helmholtz' ideas were found in his lecture notes for the summer semester 1894. He fell ill during the course, and died three months later. His notes were used as basis for a textbook by his co-workers [22; found in July 2010]. They seem to have sought additional guidance in the common literature, thereby constructing a hybridized approach consisting of incompatible elements. Apparently, they missed the novelty of Helmholtz' concept the core of which is, in fact, the only genuine precursor to this author's work [18].

Conclusions

Having now spent considerable time on learning to read early 19th century physics in its historical context, done by people who were singular pioneers, but nonetheless humans living in their time, it keeps amazing me just how many odd traps there were which had to be recognized before it was actually possible to formulate a spatial theory in an unequivocal way. The properties of vector spaces are taken as a matter of course in the geometric thinking today to such a degree that it is hard to imagine that this was something that needed to be figured out. The entire body of linear algebra would be impossible without it.

The purpose of this paper is not to accuse Cauchy of not knowing what would be discovered only much later. The task to re-evaluate the basis of a physical discipline is always the obligation of later generations. Unfortunately, this was not done in continuum mechanics. The period from 1850–1870 would have been the perfect time to search for a better understanding; instead the first law of 1847 was 'adapted' to fit on the already existing theory [19]. This led to canonization of obsolete concepts which seriously inhibited proper understanding, and in some cases to attempts to preserve Cauchy's theory against better insight [20].

Continuum mechanics and the theories of stress and deformation based on Cauchy's approach must be considered a fairly successful discipline from a phenomenological point of view, in the sense that it delivers apparently useful results in many cases. Nonetheless, the discipline has remained in most generous disregard for the advances in mathematics and physics from the mid-19th century on, be it linear algebra and vector space systematics, general tensor and vector field theory, potential theory, the discovery of atoms and bonds between them, or the physics of changes of state and the thermodynamic theory from the first law via the equation of state to the virial law of Clausius [19, equation (4a)] which is indeed an equation of state suitable for the vector field theory.

Cauchy's theory is pre-thermodynamic, historically and conceptually. Thermodynamics is based on the distinction of system and surrounding. Cauchy's approach cannot make that distinction. The work involved in Cauchy's approach in displacing points in the context of Newtonian mechanics is work done under the conservative energy conservation law $E_{\mathrm{kin}} + E_{\mathrm{pot}} = \mathrm{const}$. However, deformation work is non-conservative work in the sense of the first law of thermodynamics $\mathrm{d}U = \mathrm{d}w + \mathrm{d}q$, done by a surrounding upon a system. The Cauchy theory must cogently deliver a zero result for the work done in an isochoric deformation [19, p 4870]. The form of the first law which is used in continuum mechanics has been shown not to be the first law at all [19, p 4868]. Continuum mechanics does not have a valid work term.

The key to elasticity and deformation of solids and fluids is the theory of potentials and the Poisson condition div $\mathbf{f} = \kappa$, and the principles of thermodynamics, starting with the first law in its proper form, and an equation of state [18].

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